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ON "LEARNABLE" REPRESENTATIONS OF KNOWLEDGE:

A MEANING FOR THE COMPUTATIONAL METAPHOR

By

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The computational metaphor which proposes the comparison of processes of mind to realizable or imaginable computer activities suggests a number of educational concerns. This paper discusses some of those concerns including procedural modes of knowledge representation and control knowledge--knowing what to do. I develop a collection of heuristics for education researchers and curriculum developers which are intended to address the issues raised. Finally, an extensive section of examples is given to concretize those heuristics.

The true meaning of a term is to be found by observing what a man does with it, not what he says about it.

P. W. Bridgman

S understands knowledge K if S uses K whenever appropriate.

J. Moore and A. Newell

I. Introduction

This paper is an elaboration of some simple ideas: that thinking is a complex but understandable process; that education is an interaction with the thinking process and can influence it best by respecting the current structure of the process. My intention is to elaborate some implications of those oft avowed principles within a perspective provided mainly by the science of intelligent processes, artificial intelligence. The image of teaching and education research which derives from that perspective and its methods of analysis is quite different from most current and past trends.

The crux of my argument is this. Since Euclid, axiomatic-deductive systems have, principally by default, served as model representations of knowledge for pedagogical purposes. But while such systems which stress internal simplicity and coherence may serve useful roles for some purposes, they are not good models for understanding the learning process, much less for suggesting how to enhance it. Instead we must stress simplicity and coherence in relationship to the student's prior experience and knowledge. We must better take into account intuitive and other kinds of knowledge and knowledge processing which do not fit any known formal descriptions -- let alone an axiomatic-deductve format. Furthermore, we must learn how to bring to the surface procedural and organizational aspects of knowledge which relate to the student's specific thinking process.

But how can we, indeed, can we at all expect to flaunt the "natural formal structure" of subjects like Euclidean geometry and physics? My answer is that even for the expert scientist, formal structure is only a small and sometimes superficial part of what he knows and what we must teach. The appropriate natural structure is not a formal skeleton, but the richer structure of the functioning of that skeleton in an individual.

I conclude the paper with a collection of examples from high school and college physics and mathematics which illustrate in some detail some ways in which formalism can be put in its proper place.

II. Epistemology: A Procedural View of Knowledge

The central organizing theme of this exposition is what has come to be called "the computational metaphor." The rise of computer technology has given great impetus to the study of process in the abstract and in particular to the study of the kinds of processes which may be called intelligent. Natural language understanding and production, vision, problem solving and informal inference are examples. Out of this study naturally came a concerned look at perhaps the only appropriate "natural" model for intelligent processing, human intelligence. Thus a symbiosis between natural and artificial intelligence is begun. It is not at all surprising that concepts and theories invented to illuminate and precipitate machine intelligence seem to have a great deal to say about psychology, particularly in the

areas of the representations of knowledge and learning. As human thinking serves as a model for intelligent process, so too do theories of abstract processing and machine intelligence serve as rich sources of language and ideas for drawing implications about human processing. Thought and learning seem strongly analogous to, if not merely particular examples of the activities in complex computational processing systems. This is the root of the computational metaphor.

One very important branch of the inquiry into the computational metaphor involves the formal modelling of human learning, language abilities or other such processes. I will not be concerned with that here. Instead I wish to pursue a looser but perhaps more immediately applicable vein. I will discuss implications of some of the most crude, but in my opinion, most robust ideas which arise by considering the representations of knowledge and the styles of pedagogy appropriate for human learning in the light of computational concerns. My interest here is with structuring particular domains of knowledge such as physics and mathematics so as to be maximally comprehensible and "learnable." I begin with a crude but telling procedural epistemology.

- Classification by purpose -

One may initially classify knowledge by its purpose. If knowledge is directed externally, toward the structure of physical events or abstract relations, I will call it material knowledge. If on the other hand the knowledge is directed internally toward personal functioning and the structure of thinking itself, I will call it control knowledge. One of the prime contributions of the computational metaphor is to call detailed attention to this latter aspect of knowledge which directs organization of the thinking process. The former, material knowledge, is what one conventionally thinks of as curriculum material, the mathematics and physics itself.

Let me use a computational analogy. Consider two different programs performing the same task, say playing chess. The control structures of the two may be quite different. For example, the decision on the legality of a move may be handled in varying ways: One program may check a legal move list while the other may involve context sensitive productions which can only produce legal moves. Despite the fact that in some sense the programs both know how to play chess, i.e. have the same material knowledge (in fact, there is no a priori reason that their external behavior need be distinguishable), the organization of the process by which they exhibit that knowledge may be fundamentally different. The issue of control is when, why and how that knowledge is used. In practical terms, "teaching" each program to castle may require radically different representations of the notion of castling.

In human beings, of course, the organization of the process of thinking is not so rigid as is popularly expected of computer programs. One may well speak of this control structure as knowledge - some parts of which may be learned or forgotten, some may be quite conscious, others invisible structure, some very general and others as specific as one could wish.

My purpose in making the distinction between control and material knowledge is only to point out the importance of the frequently ignored control aspect of knowledge. In fact, I intend to emphasize the importance of the close interrelationship of these two facets: in organizing information for any processor, one must take very careful heed of the character of its capabilities. The computational metaphor leads us to look carefully at the character of human thought when constructing the form of knowledge to be presented.

Let me begin exhibiting the important but subtle nature of control knowledge with a rather simple example. Is there any difference between the following versions of Ohm's Law?

$$I = E/R \qquad E = IR$$

Formally the two are the same, differing only by a trivial algebraic transformation, but anyone who has worked with such equations will admit a different feeling toward the two.

The difference is control structure; how the parts are treated and when the whole is evoked. The formal symmetry of $A = B \Leftrightarrow B = A$ is broken by a usually unspoken control convention that the symbol on the left is the "unknown" and those on the right are the determiners of the unknown. Part of the implied control structure is that one searches for the determiners' values in order to evaluate the unknown. Note that even the word "equal" can have the same control asymmetry as the symbolic equation. Consider the clash of control structures in "let 5 equal x."

The control implications in lexical ordering are clearly almost causal: the symbols on the right taking on the values that they do <u>causes</u> the symbol on the left to have its value. $I = E/R$ is a particularly felicitous representation in this respect as it meshes well with the common causal interpretation of E as an externally established <u>impetus</u>, R as a given <u>obstructor</u> and I as the <u>resultant</u> caused by E acting against R. In $E = IR$, the interpretation of E as a result of a controling impetous I is somewhat harder to make. $R = E/I$ has the causal interpretation of Ohm's Law directly opposed by the lexical control convention, and not unexpectedly is usually relegated to the status of a "derived" equation.[1]

Anyone who has the algebraic facility to use these relations can realize their identity. Yet they are sometimes taught separately and more importantly, are often evoked separately, even in experts, for the practical reason that their control structures are functionally different. An important implication of this fact is that the transformation from one to another of these forms in a problem solving situation may signify more than a trivial change of mind state in the solver.

To better appreciate the importance of the unity of control and material knowledge, perform a rhetorical experiment. Consider an expert's understanding of, say, physics. Ask him about a concept and observe the form of the response. The paradigm, it seems to me, is that he generates a situation in which to observe the action of the concept, or generates a process which involves the concept. For example, the structure of the concept of force

usually entails an agent, a form of interaction, and a recipient. The expert may not explain that, but he exhibits that understanding. He says, "If you push on a rock . . ." (Agent = you, interaction = push, recipient = rock.) In the same way, our expert does not often reply in terms of a formal nature, neither explaining what the idea can be deduced from nor what follows deductively from it. Force is more often explained by its function as "the interaction between particles which, if known, allows us to compute motion." Less often (except, unfortunately, in the context of a "standard" physics course) does one hear precise but formal declarations such as "Force is (mass)x(acceleration)."

The reason for these behaviors is more than pedagogy on the expert's part. It is the fact that how he uses a concept, the control structure needed to utilize the idea, is as much a part of his understanding as the detailed formal structure of the idea. In other words, the disposition to embed the concept in a useful context is not accidental, but an expression of the kind of knowledge which is vital for functional knowing of the idea. It's the kind of knowing which causes the concept to be appropriately remembered and allows the expert to solve new problems with it. For teaching purposes it is unfortunate that much of this control structure is implicit.

- Classification by form -

The computational metaphor brings to light another frequently ignored fact: that the structure of process itself can be a mode for knowledge representation. Lack of attention to this has led to a skewing of educational materials toward the classical mode of knowledge, deductive or syllogistic logic, and away from more process-oriented representations. Many of the examples I discuss later are attempts to illustrate how process can be an effective knowledge carrier.

This paper is primarily concerned with these process or procedural forms of knowledge representation. Within that broad domain let me pick out two, in some sense antithetical forms which effect an orthogonal cut across the classification by purpose of the last section. The first, knowledge of procedure is little more than the name implies. It is characterized by an explicit surface structure which is step-by-step procedure. Furthermore, one expects that knowledge of procedure usually contains explicit reference to purpose and to what circumstances make the procedure useable. An example might be arithmetic which many consider (perhaps incorrectly) to be simply a step by step algorithm applied in appropriate circumstances to achieve a pre-established aim.

In contrast to the above rather shallow embedding of knowledge in procedure, one can imagine a deeper and more subtle form, which I call knowledge within process, in which the surface structure is not necessarily procedure, in which step-by-step analysis may not be appropriate (hence the use of the "process" rather than "procedure" in the name), in which the purpose of the knowledge is only evident in the control structure which evokes the process or in the function it serves; indeed, the actual subject of the knowledge may be quite invisible.

Gilbert Ryle in his essay on "Knowing How Versus Knowing That" picks a keen example of this kind of embedding of knowledge. In what sense does a hero possess moral knowledge? It is certainly not in the procedure he takes to rescue the maiden in distress. One might be tempted to say that the moral knowledge resides in a list of imperative maxims (knowledge of moral procedure?) which the hero consults. But this line is hard to defend. At the very least, the control structure which evokes the "morality list" in response to the maiden's cries rather than, say, a list of recipes has qualities quite different from "knowing that" I must do such or other. Ryle resorts to a form of knowledge he calls "knowing how" similar to the idea of knowledge within process. In essence it is a blend of disposition, style of action, and capability. The hero knows his morality in his capability and disposition to act morally. The knowledge lies solely in the style of behavior.

I have chosen a more scientific example of material knowledge within process which also hopefully emphasizes the fact that such knowledge is of vital importance to the class of knowing generally referred to as intuition.

Through one's experience with moving physical objects one acquires a good deal of knowledge about how much and what kind of force it takes to achieve a certain result. One constantly estimates or remembers the weight of objects and applies forces appropriate for what one wants done. The process is unconscious or often seems so, but vividly makes itself known when it draws a wrong conclusion. For example, everyone has had the experience of nearly throwing an empty container into the air because it was presumed full. This exhibits a knowledge within process about the relations of force, weight, and motion.

If you are posed an abstract question, "could you deflect or stop a twenty pound pendulum about to crash into you at 3 miles per hour," you presumably use that same knowledge within process. You do no particular analysis or calculation beyond that necessary to make the numbers more meaningful (say, replacing the pendulum with a bowling ball and 3 mph by 5 feet per second). It seems reasonable to describe your thinking as imagining the situation and observing your own disposition in such circumstances. Thus this kind of intuition is the accessing of knowledge within process by observing oneself.

Such knowledge about moving bodies and forces appreciates a qualitative side of the physical concept of momentum, its relation to mass and velocity, and its conservation (to the extent that inertia represents conservation). These structures undoubtedly play an important role in learning the more formal and precise physics of momentum. Later I will discuss how particularly the control structure of formal physical ideas may be inherited almost entirely from intuition. For example, force as a cause is a vital part of the knowledge within process I'm talking about, and I will discuss how this involves a control structure which is sometimes, but not at others, appropriate for doing physics.

In a less positive vein, procedural understanding of this sort can be the source of

unfortunate confusions. Measured in an image of pushing to achieve "more motion," changing energy is indistinguishable from changing momentum. That fact causes much confusion for elementary physics students (and did for Galileo as well!). In either case, productive or counterproductive, one should be aware of the possible help or hindrance from these rough but insistent dispositions.

In admitting knowledge within process as knowledge we are committing ourselves to a much richer epistemology than might otherwise be acceptable. On the surface such knowledge does not look like knowing a fact in any standard sense. Its evocation may be harder to see and its influence more subtle and more context dependent. More importantly, the ways in which past experience can serve as "knowledge" depend heavily on mechanisms available for invoking and applying it. Consequently a complete epistemology must be procedural in the sense of dealing with the processes of interpretation, analysis, transformation, etc., which can cause knowledge, particularly rather invisible forms like knowledge within process, to have significant influence in learning or problem solving. I turn now briefly to this area, accessing knowledge, specifically in the context of accessing knowledge within process as in the above example of physical intuition.

- Accessibility -

One reason that knowledge within process is so frequently ignored and not exploited is that it is almost always non-verbal. Even more than that, it may be "inaccessible" in the sense that the student himself does not realize that it is knowledge and can be put to use.

A similar inaccessibility can be still more devastating in the <u>teacher</u>. Consider: An instructor stands before the blackboard and declares that some problem obviously should be approached using Newton's Second Law. A student says that he did not think to do it that way; he says he doesn't understand. The teacher again declares his approach obvious and as justification proceeds with the details of the solution. But the student is no better off than before. The student was asking for the control knowledge which brought Newton's Second Law to the teacher's mind, not the post hoc verification - "see, it works." There is something that the teacher "knows" about $F = ma$ which the student does not know, something which evokes the law in the face of a certain class of problems. Optimally the teacher should know why he thought of that method and be willing and able to discuss it. Otherwise the student may or <u>may not</u> generate the appropriate understanding. This is a clearcut case of control knowledge encoded within process (in the teacher, not encoded at all in the student).

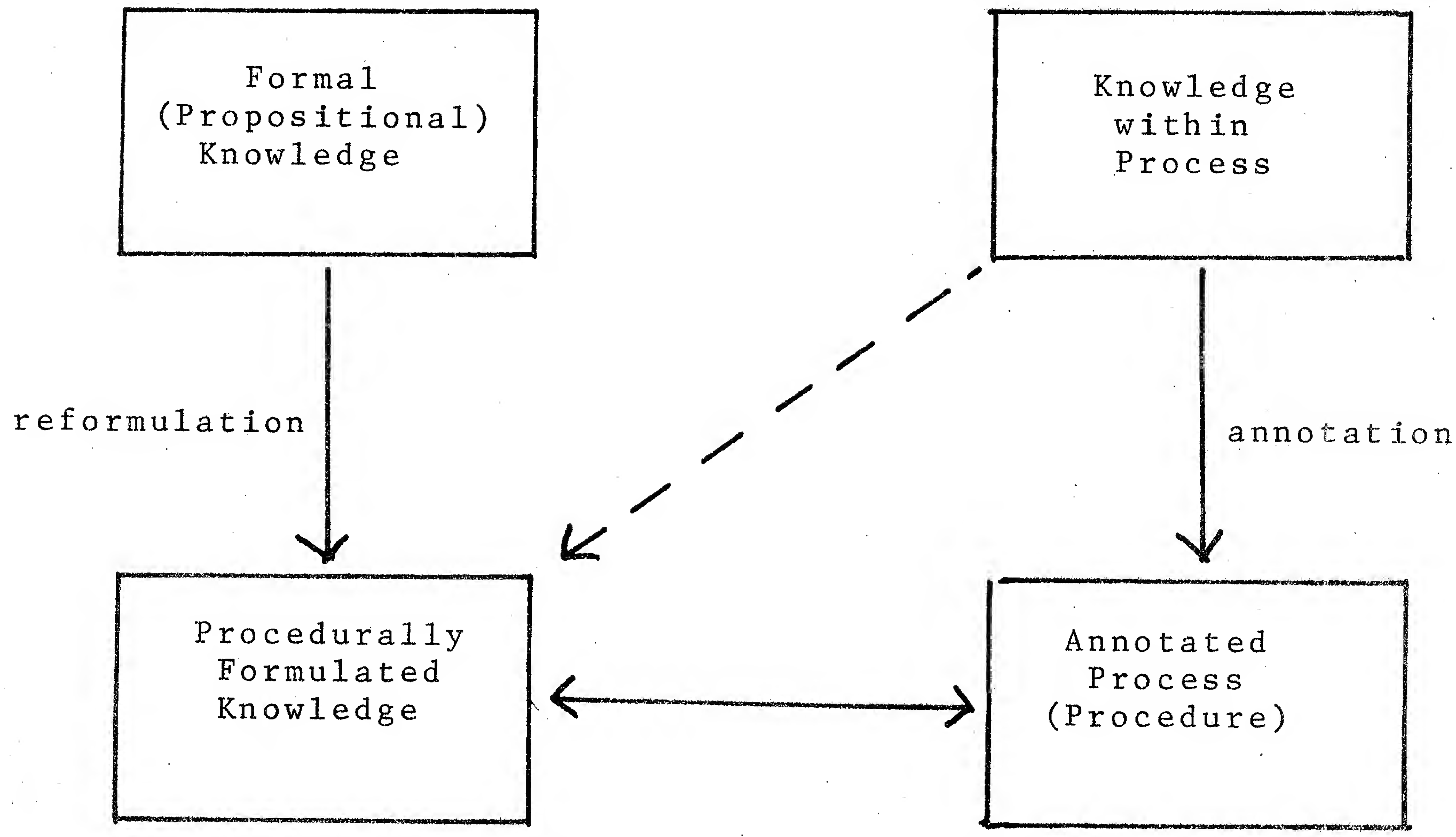
- Annotation -

A fairly general process for accessing[2] knowledge within process is <u>observation, analysis, and annotation</u> (hereafter referred to as annotation). With help from a teacher or other source a student can study his own behavior (or some other negotiation of circumstances) in a particular situation, trying to understand its detailed purpose, what it produces and how it succeeds. Clearly selection of the situation and guidance in what is relevant are important

functions of the teacher (or education researcher). The student identifies functional parts of the process, naming them appropriately, making connections and adaptations to a formal scheme or other relevant knowledge. The end result may well be a completely annotated version of the original process, the top level of which may indeed be the sort of knowledge I have called knowledge of procedure. In other cases that kind of product is not relevant; one may merely be striving to make an explicit and useful connection between formal learning and experience.[3]

It is worth remarking that a fully annotated version of some knowledge within process can be invaluable in teaching. If Ryle's hero were to teach morality, he would undoubtedly compile a list of maxims which annotate his dispositions in imperatives. To cite a different example of the possible roles of annotation, it seems clear to me that the learning of noun and verb classification and grammar in general should be very much a process of annotating one's dispositions toward word use rather than learning to apply an abstract criterion. A noun is the <u>way</u> one uses words like cow or apple in speaking with respect to concerns like word order, the possibility of acquiring modifiers in a certain way, etc.; the description as "a person, place or thing" does not capture any reference to students' own knowledge within process about generating or understanding grammatical sentences. In a parallel vein defining words, especially determining shades of meaning between similar words is very often annotation in answering questions like, "how would I use this word; what would I think of if someone said the word to me; can I describe the image the word conjures up?"

Below is a schematic of a possible path of contact between knowledge within process and more explicit (propositional) knowledge. The dotted line indicates the likelihood of direct inheritance of control knowledge by procedural formulations. See the discussion of force as regards causal syntax and momentum flow in **Examples**. The **Examples** section will also carry the burden of conveying the notion of procedural formulation.



The following diagram is a summary of the procedural epistemology presented here and contains further examples of the classification.

	Material Knowledge	Control Knowledge
Knowledge of Procedure	Specialized Processes -Addition -Differentiation	Heuristics Mnemonic Strategies -Knowing the "One is a bun" strategy for remembering lists. -Knowing to "Look for simple or extreme cases."
Knowledge within Process	-Intuitions of what is true (e.g. a mathematician's "estimate" of the truth of some theorem). -Knowing how to walk a straight line. -What one accesses to estimate the effect of a blow on a moving object. -Knowing that April has 30 days by annotating a recitation of "30 days has September ... "	-Intuitions of what to do or "reflexes" like hitting the brakes in a car. -Organizational productions as invoking convenient verbal or pictorial conventions (even the disposition to draw pictures or verbalize!). -"Thinking of" the right specialized procedure to use in a physics problem.

In closing this section I caution again that this epistemology is neither exclusive nor exhaustive. It is only to serve in identifying some important computationally relevant points concerning knowledge representation and acquisition.[4]

III. Against Axiomatics

To motivate and clarify the import of the computational metaphor I wish to develop contrasting implications of another, much more common image of knowledge representation, and by extension, another metaphor for the educational process. At the risk of oversimplification one can call this other image the logical formalist (e.g. axiomatics and deduction) metaphor.

In what follows it must be understood that I am using "formal" in a special way. In particular I am using axiomatics to represent the class of formal representations in characature as a style of knowledge representation which strives to remove the subject material as far as possible from the confusions and errors possible from using unannotated structures like intuition. Unfortunately such abstract presentations frequently underestimate the amount and kind of knowledge, particularly in the domain of control knowledge, needed to make the formal system function in the student. "Informal" is, then, by contrast an attempt to use structures the student has, annotated or not, change and debug them rather than just write them off as a lost cause. But let me start from the beginning.

A prevalent and influential view of the educational task involves dividing the learning process into a large number of discrete steps, the logic for each of which is impeccable. This image closely parallels and perhaps even derives from the image of mathematics (or science) as deductive systems based on a small set of axioms or laws. As each step in the construction of a mathematical theory is to be infallible in its careful and precise deduction from previous levels, so each education step is simple and easily learnable by the student. The knowledge acquired from his step by step edification is supposedly as secure as the collection of theorems comprising the finished mathematical theory.

There are several inadequacies of this view which stem mainly from implicit assumptions about the nature of human processing.

- Misconception 1: Science is deductive -

The emphasis on the axiomatic system as an end result of mathematical development unfortunately has led some to pick this out as a principal characteristic of mathematics. But what mathematicians <u>do</u> does not have the character of the formal endproduct. Below, R. Courant speaks to this point in the context of eighteenth and nineteenth century mathematics history.[5]

> "In Greek mathematics we find an extensive working-out of the principle that all theorems are to be proved in a logically coherent way by reducing them to a system of axioms, as few in number and not themselves to be proved. This axiomatic method of presentation, which at the same time served as a test for the accuracy of the investigation, was ... regarded (then and still today) as a model for other branches of knowledge ..."

> "But it was a different matter with modern mathematics ... In mathematics the principle of reduction of the material to axioms was frequently abandoned. Intuitive evidence in each separate case became a favourite method of proof ... Blind faith in the omnipotence of the new methods carried the investigator away along paths which he could never have travelled if subject to the limitations of complete rigour ..."

"In spite of all its defects, intuition still remains the most important driving force for mathematical discovery, and intuition alone can bridge the gap between theory and practice."

Morris Kline continues the argument for me.[6]

"Neither Euler nor Gauss could have defined a real number, and it is unlikely that they would have enjoyed the gory details. But both managed to understand mathematics and to make a "fair" number of contributions to the subject. ... most teachers, instead of being concerned about their failure to be sufficiently rigorous, should really be concerned about their failure to provide a truly intuitive approach."

Even in physics it is very tempting to regard the deductive and analytic development of mechanics from the cornerstone F = ma as archtypal physics. Yet this is hardly the case. Certainly Newton in his Principia exhibits no sense of following a coherent linear deduction from secure basic principles. Historically in almost all physical theories which had more than private and personal (hence not very accessible) developments, it is clear that a major part of the effort required to build them was spent in struggling at a heuristic, imprecise level. Let me cite the case of perhaps the greatest achievement of twentieth century physics. Though quantum mechanics is formally a theory of vector spaces and linear transformations, Heisenberg when he began the theory did not even know what a linear operator or matrix was! But he knew that he needed to know and searched out mathematicians to find out!!

It should hardly be necessary to point out that the historical lesson applies as well to the present. Even in this day when the image of science is in essence precision, there are no precise, deductive or axiomatic landmarks at any of the frontiers. It is as evident that future mathematical developments will be based more on new definitions and axioms than they will be simply following the road laid down in any current deductive scheme.

The lesson is that the intellectual mechanism of scientific creativity and discovery is structured very differently from a formal synopsis of results. It is a mix of guessing, heuristic and refabrication built on established but very different ideas than those ideas which constitute the end results. I suggest that image is much closer to what the process of learning in general must be about than the simple model of being fed the fundamental principles which contain via a transparent mechanism (say, deduction) the whole "knowledge."

Is it not reasonable to assume, therefore, that a major part of schooling must involve students in situations where they must carve out larger chunks of knowledge, perhaps developing a personal mini-theory, rather than to merely prove sublemma x, given y, or apply physical law z to this situation? If we do not go that far, should we not at least

present material in a non-axiomatic form which can develop tolerance for and abilities to use ambiguities, appreciation for guiding principles which are not infallible and other skills which are not usually addressed in "closed" situations. Shouldn't one spend effort on knowledge and skills at a level above the intellectual mechanism of deduction, the level where the important first steps in problem solving take place?

- Misconception 2: Deduction is simple -

It is a great temptation to assume that axiomatics and deductive processes are the simplest building blocks of general intellectual skills. Axiomatics do have a simple step-by-step nature but only within a formal logical framework and assuming an overview which provides a general direction toward what are to be considered the final results of the system. Regarding the logical framework, it must be understood that early students are as much trying to understand and appreciate that framework and its intent as to learn specific mathematics such as geometry. Theorems, axioms, lemmas, even definitions[7] along with countless proof strategies all have intent and meaning which is invisible to the beginner in the subject. All too often the result of ignoring this is an alienation caused by feeling pressed into following rules of a game which are arbitrarily made up by someone else and which are not at all subject to discussion or argument. Furthermore, the understanding of the "natural" flow from axioms to theorems is entirely obscured within the aura of formal infallibility if the axioms are not felt to be secure, or alternatively, if the theorems seem as obvious as the axioms[8] -- two phenomena which are common particularly in elementary geometry. I think it is unfair to insist students follow this flow without the feelings of necessity and security concerning the axioms which were generated in the head of the person who made up the deductive system only after considerable experience in the domain on a tentative, heuristic, and perhaps even playful level. (Euclid's accomplishment was after all organizational, the "content" was previously established, mostly by others, outside of the axiomatic form.)

The mismatch between deductive systems and the character of human thought has surfaced strikingly in current artificial intelligence work. As it has become more and more clear to artificial intelligence researchers that deduction is simply an intolerably poor model of human thinking, so should it become clearer to educators that deduction is inappropriate as a general model for pedagogy. The interested reader will find a more complete treatment of this in [Minsky 1974].

I am not arguing that axiomatics should not be taught, but that they should be approached in somewhat the same way that the active mathematician or scientist approaches them. Axioms and other summary organizations are in order after and as a result of searching for precise and foolproof understanding of the sort that intuition and informal reasoning are not.

IV. A More Human[9] Style of Pedagogy

I am advocating that abstract formalism be avoided as a basis for, or model of curriculum, especially on an elementary and introductory level. Instead we need to find a more human style, one that is more intuitive in the way it accesses and generates non-formal knowledge; one that is more heuristic in that it has specific concern for control knowledge; one that is more informal in appealing to other levels of justification for its reasoning than just axioms and deduction.[10]

- Teaching what to do: material knowledge -

To begin I must make a remark about content. The remark is considered obvious in many contemporary educational circles, but bears repeating. When one asks the fundamental question, "what is it you want to teach?" the best answer is not "science" or "math" but "what scientists (or mathematicians) do."[11] The point is to shift emphasis to activity and away from facts. After all, it is no doubt more correct to say mathematicians know how to generate proofs for x or y than that they know the proofs. Physicists generate solutions to problems; they don't know them. It is not that facts (or even calculational algorithms) are irrelevant, but that the higher level activity of deciding when to use a fact or invoke an algorithm or invent a new algorithm or look for a new fact is more characteristic of scientific knowledge than the particular set of facts involved. This of course is not surprising as doing new things is precisely the raison d'etre for scientific knowledge. In other fields even as far removed as the arts one sees clearly that the skills and abilities to reshape the old in the face of a new context better characterize the successful practitioner rather than just ability to recall the old.

The implications of the above are two-fold. First is the direct realization that much of what is to be taught is procedural in quality. On a rather primitive level this is recognized in current curricula at all levels. Algorithms from reading to adding to chi-squared are stressed. Unfortunately this does not extend to higher level understandings which might teach children, for example, to reinvent "carrying" if they forget the precise mechanism.

Secondly, even facts should be taught with connectives to procedures; what a fact means must be functionally clear in how it can be used as an input to established or even possible procedures. The lesson taught by Einstein in the early part of this century is a model for this declaration: numbers are meaningful only as the product of a particular measuring process, not in their interpretation within an a priori scheme. Concepts, it goes without saying, are as subject to this mandate as are facts. It is vital to know the proper contexts and functioning of a concept to appreciate its relevance and to use it effectively.[12]

- Multiple representations -

There is implicit in this stress on relevance and connectivity of knowledge an urge toward multiple representations of the "same" knowledge. Concepts are very sensitive to the process

context of their use. Accordingly, one expects that the general rule will be a multiple faceted approach rather than an attempt to capture all context-related possibilities in a single definition or axiom. It may seem that this means much more to teach. But in the end a coherent but large structure will almost certainly win out over a small but obscurely dense presentation. A poem may easily be memorized while a terse encoding of the same ideas in a sentence of nonsense syllables can be "unrememberable."

The fundamental assumption behind this idea of multiple representations is that a rich, overlapping collection of different views and considerations is much more characteristic of preciseness in human knowledge than a small, tight system. In terms of problem solving the claim is that the parity of restatement or translation is as or more important to problem solving itself compared to the hierarchy of deduction.

- Micro-skills -

I do not wish to be misunderstood as saying that high level, large scale procedural knowledge is all that is important. There are certain "micro-skills" which simply must be mastered in whatever way the student selects. Perhaps the most vital "knowledge" of trigonometry is the simple skill of quickly and correctly identifying the components of vectors. A student who can prove angle addition formulas and all the rest will truly have gained little from trigonometry if he takes several minutes to find the hypotenuse of a right triangle given an angle and one of its sides. Typical of knowledge within process in general, one does not often find educators cataloging or doing analysis of the many sorts of "compiled" micro-skills which are associated with specific curriculum.

- Teaching what to do: control knowledge -

The recognition that the activities and procedures we are discussing take place within intellectual structures has other implications. First, directing one's own activity must be a prime target of the educational task, even if it is not explicitly addressed. One must provide an answer to the question, "What do I do now?" Secondly, the procedural content and procedural relations of facts and concepts must respect the intricacies of personal functioning. They must contain information, implicitly or explicitly, about what is easy and what is hard to do with human mental machinery: how to use that machinery efficiently. This returns us to heuristic and intuitive levels of understanding for there can be no doubt that these are effective yet thoroughly human and personalized levels of processing. The feelings that many teachers and educators have that such concerns are imprecise and irrelevant to the "real" material must be overcome.

- Taking advantage of what is known -

The movement away from axiomatics has the bonus of not only developing knowledge and skills which are more "human," it allows one to tap knowledge and skills which already exist but are usually unused in formal contexts. To mention two, practical language skills and the active, intuitive geometry which allows one to navigate and physically manipulate the world are both ignored in explicit formal treatments. These do not have deductive support but

nonetheless are secure due to the rich experience gained in dealing with the practicalities of the world. Being able to choose and walk a straight path across the room is every bit as much knowledge (albeit knowledge within process) about geometry as Euclid's axioms. I will have much more to say about this personal and active geometry. For now suffice it to note that it is quite far from the static geometry which is usually taught as the first step into mathematics after arithmetic.

Beyond such specific knowledge which a less formal approach is meant to tap, I would like to argue that axiomatics cannot engage the more general style by which people quickly and effectively learn about the world. People are more fundamentally model builders than they are formal system builders. They reason by analogy. They induce. They formulate heuristics and develop dispositions to act in certain ways in certain circumstances. Their views are as much conflicting patchworks as they are coherent systems. Yet, despite all the non rigor, they learn a great deal about the world and they learn it well in a functional sense. A junior high school student learning axiomatic geometry must throw all this well-practiced methodology out and restrict himself to the most meager of learning styles.

Developing in the style of and from the contents of heuristic and intuitive world models is a very robust structuring of knowledge. Though personal "world models" may be mistaken in many details and may seem on the surface quite imprecise, they are secure in that they do not arise from abstraction but from procedures which work. Not only is it very likely that these ideas generated from experience contain germs of truth or senses of interpretation which are valuable, but the student knows this. He can have a confidence in ideas which come from knowing that he can navigate the world though he does not know theorems in Euclidean geometry. Again, geometry will be a major example when we turn to explicit examples.

- A dynamic curriculum -

Experience-based knowledge is also robust in relation to its extensibility. Here the diffuseness in the statement of questions and problems characteristic of informal understanding again pays the same dividends to a student seeking to extend and explore on his own. It is much more likely that one can dig out meaningful and precise understanding from imprecise ideas suggested by informal inference from real world experience than to expect a beginning student to be able to suggest productive new lines of enquiry from formal similarity or other such operations inside a deductive system. A well-structured curriculum allows students to ask interesting and productive questions rather than waiting for the next exercise. It is a rare child who will propose, let alone prove his own theorems in Euclidean geometry. To return to an earlier argument, if the aim of mathematical education is to involve the child in mathematician-like activities, it surely seems reasonable to provide him an environment much richer in suggestions and pointers to new enquiries than the sparse combinatorics of axiom juggling. A rich yet fuzzy intellectual environment can in itself return full circle to provide the appreciation and feeling for the proper use or axiomatics and thus provide motivation and sustenance when the time comes that careful

formalism is necessary.

This richness we speak of in an informal environment should well be considered a goal in itself. To use an image, one wants a world as rich and ambiguous (in the sense of many possible connections between parts) as an erector set or a good set of blocks, not the dull and sometimes frustrating efficiency of a prefabricated toy model which has been cleverly designed to go together in only one way. A rich environment is dynamic in its interaction with an individual.

- Heuristics -

There are three final notes to make before turning to examples. First of all, though there has been much talk of heuristics in this paper I do not mean to say that the program here is merely to present students with a thorough and complete list of general guidelines for solving problems. Such an approach as taken by Polya may be useful, but it cannot be the whole story. What I am arguing for is that curriculum material be organized with great concern for the control structure in the student. Heuristics, informal statement, analogies, etc. should all be an integral part of the material taught. These sources of control knowledge should be grounded as far as possible in the knowledge already present in the students.

- Holes and windows -

To be sure, one will have to tolerate in this curriculum a great number of "holes." The student will have to deal much earlier with the realization that he (and his teacher!!!) has only partial understanding. But the illusion of perfect knowledge fostered by axiomatic presentations is well buried. As a great return from the gaps in the students forming knowledge, the ground beyond that which has been well staked out will not only consist of holes devoid of comprehensibility but, sometimes, windows through which further and perhaps more profound understanding may be seen.

- An intuition-formalism truce -

Again, I have not written symbolic or any other kind of formalism out of this pedagogical theory. Certainly such intellectual machines play an important role in summarizing material knowledge and insuring a uniform precision. But I have argued that it is a great mistake to identify knowing a field with knowing a formalism.

The examples which follow are intended to point the way toward formalisms which are good physics and mathematics as much as they are intended to reflect cognitive realities. There is no reason a learnable curriculum cannot allow students thorough precision when that precision is the content of the matter.

V. Examples

The problem of developing curriculum following the above arguments for concern for intuition, for concern with control knowledge, for experiential support, is non-trivial. At the very least, the question of inculcating in the student an important disposition (the inverse operation to making knowledge within process explicit) is problematic. We cannot merely insert into the student's mind a "demon," as one may figuratively do in a computer program, which activates itself in appropriate circumstances. Nor can we expect practice as the mindless rehearsal of musical scales or arithmetical operations to succeed very well in such complex and constantly changing situations as doing physics problems (even if we could set meaningful examples to practice!). But we should explain carefully to the student what we understand of the disposition, its cues and modes of operation. Furthermore, and probably more importantly, we can organize the material around and incorporate into it the relevant features to the final dispositional encoding. It seems likely the student will always have to make that last step for himself.

To aid in this broad program one has the "new" outlook of searching for procedural formulations rather than air-tight formal ones, but the roadway ahead still is by no means clear. This section is an attempt to sketch some examples of the kind of reorganization envisaged. These hopefully can fill out and support the rather abstract discussion in earlier sections.

Physics is a protean ground for exploitation of procedure and intuition. It is by its nature mechanism. There is, furthermore, prima facie evidence that children have the sort of procedural-intuitive understanding of physics that we can rely on to replace deduction and propositional security. A child can catch a ball thrown in the air and knows roughly what will happen when you push on things in most circumstances. Below is a sketch of some ideas concerning mechanics which embodies the concerns we have been discussing.

1 - The Concept of Force

What is the concept of force as usually taught in elementary physics courses? After a few philosophical remarks about pushes and pulls, there appears $F = ma$. From there on, by and large, discussion of force revolves around this analytic representation of the idea. Operational understanding is for the most part left to the student to construct out of many workbook problems and examples of expert solutions. I would argue for explicit treatment of the precise mechanism that force is and for greater effort to bring to bear intuitive knowledge.

What does force do to a body? Clearly it changes something -- not the color nor the taste. To zero in on what force changes, it makes sense to consider a situation where there is only one force of significance and one simple object. Perhaps a hockey puck on ice is a good example. What happens when you push on it? If it is standing still initially, the puck goes in the direction of the push. Does force then change position? Certainly not -- at least not exactly. A moving hockey puck subject to a push does not simply go in the direction of the

push.

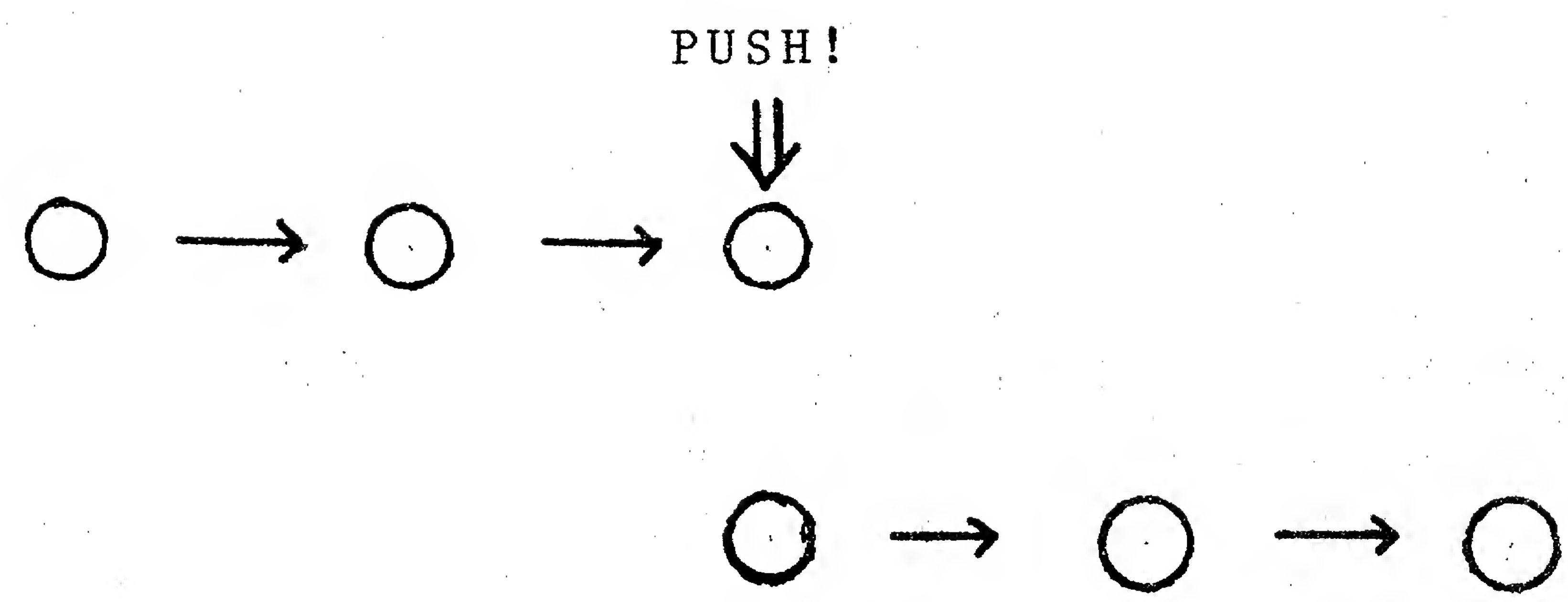


Figure 1. Force changing position.

The above does not happen, but rather:

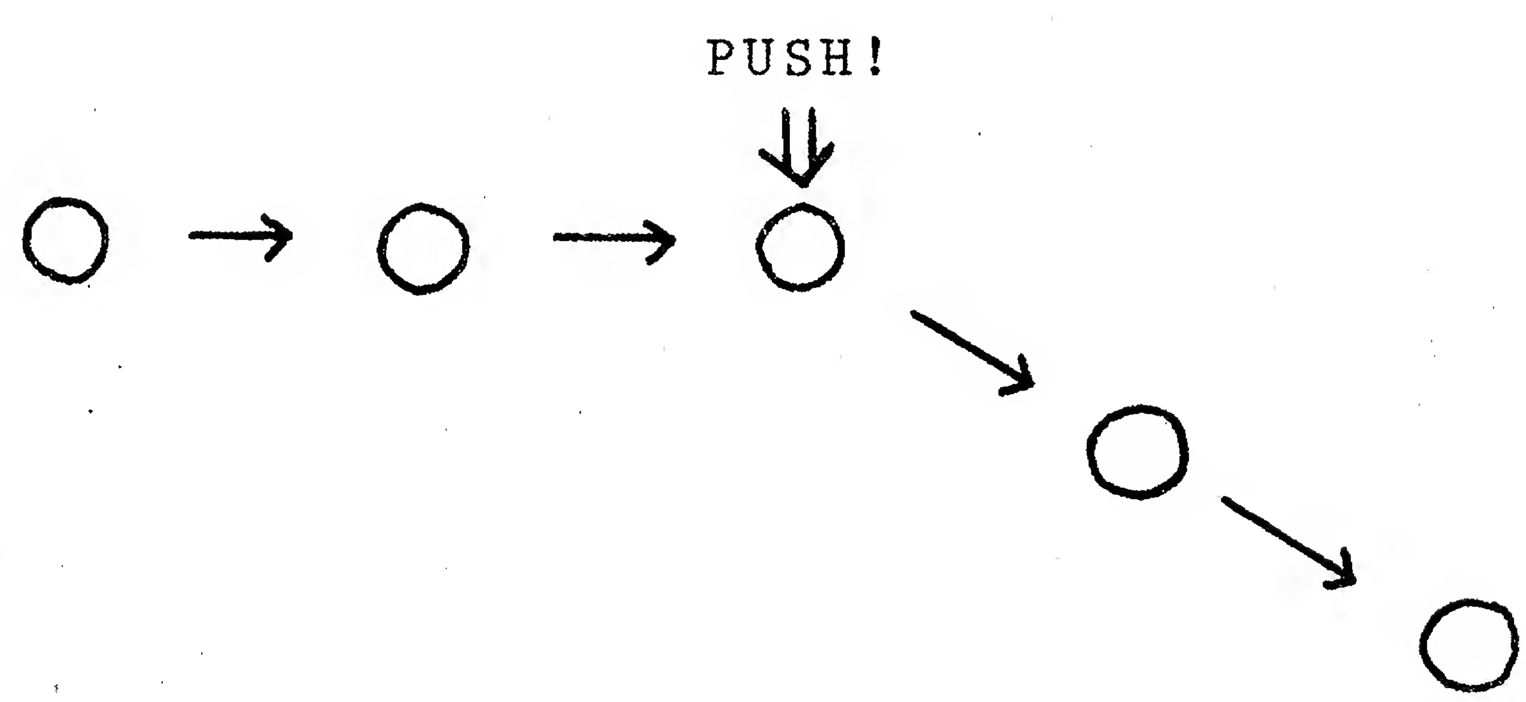


Figure 2. Force changing velocity.

Force deflects, it does not move. A more precise way to say that is: force changes the velocity (direction included) not position. It is the sole function of force to change the velocity. This parallels the function of velocity which is a "changer" in its own right. Its responsibility is to change the position of an object.

Thus we have arrived at a remarkably simple yet precise picture of the mechanism of motion. Velocity is the sole changer of the position, and force is the sole changer of velocity. That is not obvious, but it is true and is only a small (but important) step away from the naive "push or pull."

There are immediate consequences for these ideas. For example: (1) The "counter-intuitive"

action of a gyroscope can easily be understood if one looks to see how the velocity of the wheel parts is changed rather than assuming pushing something makes it go in the direction of the push.[13] "Push or pull" is not adequate qualitative knowledge about force to account for gyroscopic action. (2) Given an analytic representation like vectors for velocities and changes in velocities, an algorithm for generating particle motion is immediate. Roughly speaking, if $\vec{v}$ = velocity, $\vec{F}$ = force and $\vec{x}$ = position, each particle is continually computing its next position and velocity by

$$\vec{v} \leftarrow \vec{v} + \vec{F}$$
$$\vec{x} \leftarrow \vec{x} + \vec{v}.$$

That is essentially all there is to a very concrete manifestation of this understanding, a computer program to simulate any "forced" motion. It is a formalism in the best sense which stands in the proper stance with respect to intuition and qualitative understanding, flowing naturally from them but at the same time refining them. (3) A particularly good example of the simplicity of this point of view is gravity. All those complicated trajectories are the result of the simplest of all possible forces. One that acts equally on all objects all of the time. In the above equations, $\vec{F}$ is a "universal gravitational constant."

- Bugs and refinement -

Now of course there is some refining to do. If F is not an impulse force, but some continuous "pouring of velocity" into an object, one should write

$$\vec{v} \leftarrow \vec{v} + \vec{F}\Delta t$$

where $\vec{F}$ is the amount of velocity per unit time which is added to the initial velocity. This matches the more explicit

$$\vec{x} \leftarrow \vec{x} + \vec{v}\Delta t.$$

Furthermore, to make this notion of F coincide more closely with the intuition of force as effort one must realize that heavier objects require more "force" to make a given change of v, thus one has

$$\vec{v} \leftarrow \vec{v} + (\vec{F}/m)\Delta t.$$

This refining I do not take to be a disadvantage. Heuristic information is often characterized by a hierarchy of ideas with the key ideas on the top and successive lower levels of warning, restrictions, corrections, just as any procedure has a global plan but also many conditional and contextual parts. "Oversimplification" with successive corrections is, I think, as much a good mode for presenting much curriculum material as it is a general workhorse method for computer programming. It is not in any way intended to be a sloppy understanding, but it is meant to be an organization which allows one to (1) keep the information most necessary for actions such as problem solving on top, and (2) allows one to choose the top level in such a way as to be tied to intuition appropriately. By these statements I do, in fact, mean to imply that force as a velocity changer is both a key idea in problem solving and an idea with considerable intuitive support. On the other end, the notion of "refinability," that the top level heuristic and qualitative understandings must develop naturally into more precise and careful treatments, is important to the approach.

- Causal syntax and force -

Let me take another look at what may be happening in the student's head when we introduce Newtonian Mechanics. I implied earlier in an example of knowledge within process that the causal structure - agent, interaction, recipient - is important to the concept of force. For reference I will call this triple causal syntax. It is important in the first place because it is a naturally occurring structure in terms of which humans will interpret similar structures such as the notion of force. The personal context of force is almost always a causal situation: I cause something to move, or it causes me to recoil, etc.

The recognizable form, causal syntax, gives an interpretation of the newer formal concept of force in more primitive terms. At a deeper level there is a control structure implicitly attached to the causal syntax which is appropriate for force. One always assumes a reasonable connection between the causing agent and the resulting actions of the recipient. It is therefore appropriate to try to establish the result of the agent's initiative given a suitable description of it. It is also appropriate to try to infer specifics about the cause, given the causal mechanism and some specifics about the effect. These translate into dispositions to calculate motion from force and infer force from motion.

There are also negative dispositions obtained from the natural identification of causal syntax in the concept of force. The syntax is directed, not symmetric. One is not disposed to treat the agent in the same way that one treats the recipient. The earth causes a ball to fall, but one does not think about what effect this has on the earth. Elementary physics students are notorious for thinking long and hard about the force B exerts on A having just pronounced the force A exerts on B. Though the symmetry is always taught in Newton's Third Law, "equal and opposite reaction," almost never does one see a careful, explicit confrontation of that proposition with the intuitive causal structure of force. As a consequence of that, the proposition of symmetry languishes as a formal idea devoid of appropriate control structure until the student has many times been chided to remark on its implications.

- Multiple representations: force as momentum flow -

When one looks at statics problems instead of dynamics the concept of force as velocity changer is almost empty of intuitive content. We must restructure the concept of force unless we intend to let formal analytic implications of $F = ma$ carry the burden (as they are capable of doing, formally). To this end one can introduce the concept of momentum, mv, and force becomes a changer of momentum. The reason for this is that one has more structure; momentum is a conserved quantity. Thus force is not just a change in momentum of some particle, it is a transfer from one particle to another. Providing one can invoke strongly enough the image of momentum as a conserved entity, just as water in everyday experience, a whole host of intuitive knowledge (such as "what goes in must either come out, or it collects!!") becomes available for use in problem solving. This image of force introduces an activity into the world of statics which is much closer to a physicist's view of a constantly working and processing world than the name statics implies. It is an image much

closer to common experience than formal equations of equilibrium.

Gravity is a force, hence it is continuously pouring momentum into the ball in my hand. The momentum must be going somewhere, I do not see it collecting in the ball (you can always see momentum collect!). The momentum must be leaving the ball, flowing through my body and into the earth. All along the way there are forces, stresses, expressing this flow of momentum. Flow of momentum is a precise and correct replacement for the "common sense" feeling for "transfer" of force through a static member. Again I am claiming the intuitively tied image will help in problem solving. Consider the following problem:

- A problem -

A train consisting of an engine and several identical cars is accelerating at a constant rate on a flat stretch of track. Neglecting friction what is the tension in the linkages between cars.

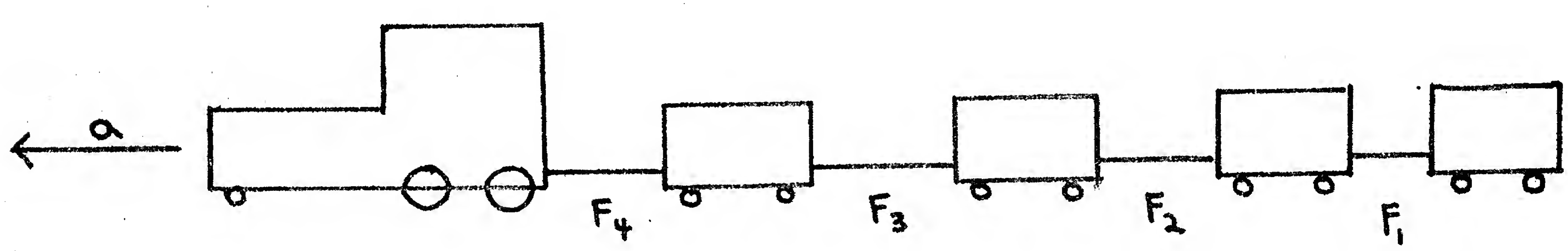


Figure 3. An accelerating train.

The standard approach to the problem directs isolating subsystems and writing down F = ma for each. The control knowledge of how one selects an appropriate subsystem is rarely if ever discussed, but let us suppose the student does the obvious, writes F = ma for each car.

$$F_i - F_{i-1} = ma, \; F_0 = 0$$

At this point for most students the equations stand bare -- the physics is done and the question of how to solve them is an algebraic one. The clever student notices quickly that the equations can be solved recursively from $F_1 = ma$; one less clever muddles through.

I have frequently encountered students not quite steeped in the system--F=ma process whose intuitions suggest a rather different analysis. They explain a feeling that each car is "absorbing" a force (of ma), leaving the car to the rear with "less force." Unfortunately the analysis cannot proceed if the only Newtonian paradigm is the F = ma one. I am about to point out how force as momentum flow can provide a precise Newtonian frame which allows this usually disallowed intuition.

The momentum flow analysis says that each car is receiving momentum from the car in front, collecting momentum at the rate of ma, and losing some momentum to the car to the rear.

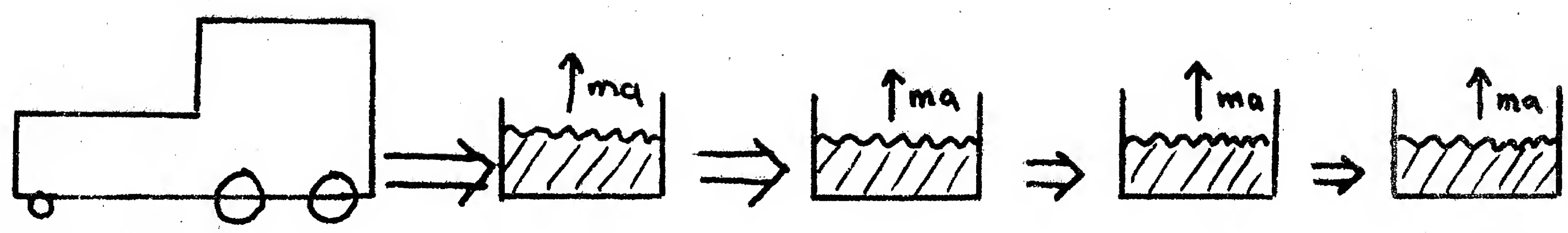


Figure 4. A momentum flow view.

The equations corresponding to this view are identical, but the sense of mechanism provided makes for an enriched possibility for attaching an intuitive control structure to them.

(1) The recursive solution method is natural in this framework. The linkage to the last car must be carrying the whole of the momentum collecting there, i.e. ma. Knowing the flow out of the next-to-last car, the flow into it is determined, and so on.

(2) But an equally transparent solution exists. Each link is providing all of the momentum collecting in the cars down stream. Thus the flow through the n^{th} link is nma. Notice that this solution corresponds to selecting a new system decomposition in F = ma terms, one which in the absence of a sense of the mechanism often poses difficulties like, "How can you just declare a collection of things a system?"

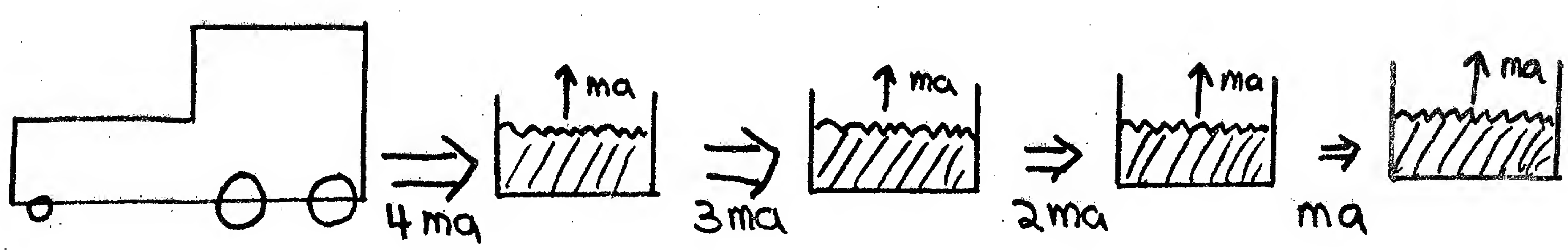


Figure 5. The flow solution.

I emphasize that the criteria for flexibility of control structure and "naturalness" of solution method are not abstract, but rely in two ways on students' previously acquired knowledge. First, we are relying on students' abilities to easily provide control analysis within a flow interpretation. Heuristics like "look upstream or downstream to see if there is a better place to measure flow," are engaged. Secondly we are relying on the fact that momentum flow is a

precise Newtonian annotation for the previously noted intuitions about "absorbing" and "transferring" forces. Thus it will be naturally evoked in problem solving situations.

The difficulties of introducing the flow of a vector quantity in analogy to fluid or other flow in order to tap experiential control knowledge do not make it a clear pedagogical winner over pure F = ma. But I argue that these concerns suggest a more coherent experimental effort in this direction than is evident in the vast numbers of "standard approach" textbooks.

- Causal syntax revisited -

I conclude the example of the concept of force by returning to the role of causal syntax. I previously pointed out that the difficulty of attatching causal syntax to the notion of force is that the syntax is directed whereas there is no physical way of separating or distinguishing action from reaction. It is facile to point out that force as a transfer of momentum is a much more symmetric image (the recipient gaining p is the same as the agent gaining -p) and to thereby conclude that it "fixes" the asymmetric disposition of the causal syntax. Indeed what is more likely is that students will find their whole feeling for the notion of force sliding away from their grasp when an essential (to intuitive understanding) primitive structure as the causal structure is threatened. Again it seems the most rational course to confront the problem directly with a discussion centering on metamorphosis of causality from the simplistic causal syntax to a more appropriate notion.

2 - Turtle Geometry

Standard Euclidean geometry approaches start with objects, points, lines and angles with which students have some familiarity, but immediately and usually quite thoroughly cut away the usefulness of that familiarity with formal axioms and the insistence on (more or less) formal proofs. Points and lines are humanely undefined, but angle is defined by a pair of "rays" and things get rapidly more abstract. Papert some years ago suggested a new sort of geometry which begins by replacing this formal angle with the heuristic, "angle is a turn." The model can be a creature called a "turtle" who knows angles by turning through them. The turtle is given mobility by being able to move in a straight line (forward or backward) when he is not turning. Turtle geometry is the study of the figures constructible by such a creature and is evidently a much more active study than standard Euclidean geometry. An elementary student can easily "play turtle" to bring to bear his own intuitive knowledge about space.

To compare turtle and Euclidean geometry on common ground let us prove that the sum of angles in a triangle is 180 degrees. The usual Euclidean method involves bringing the three angles to a point to see that they sum to "opposite rays." The problem is that doing this requires the construction of a line, l, parallel to the base of the triangle, a non-obvious activity, followed by identifying alternate interior angles.

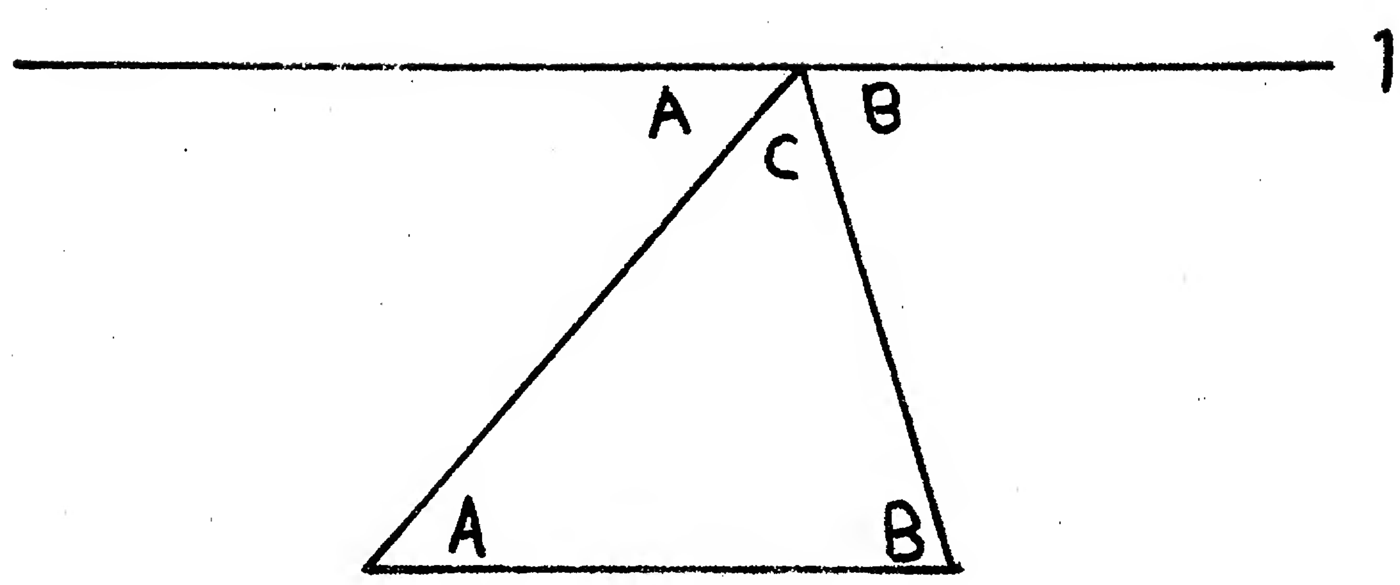


Figure 6. Proving A + B + C = 180 degrees.

To use a turtle to do the same thing, simply embed his basic angle measuring facility in a process which sums the angles: have him turn each angle in sequence in the same direction and see the result.

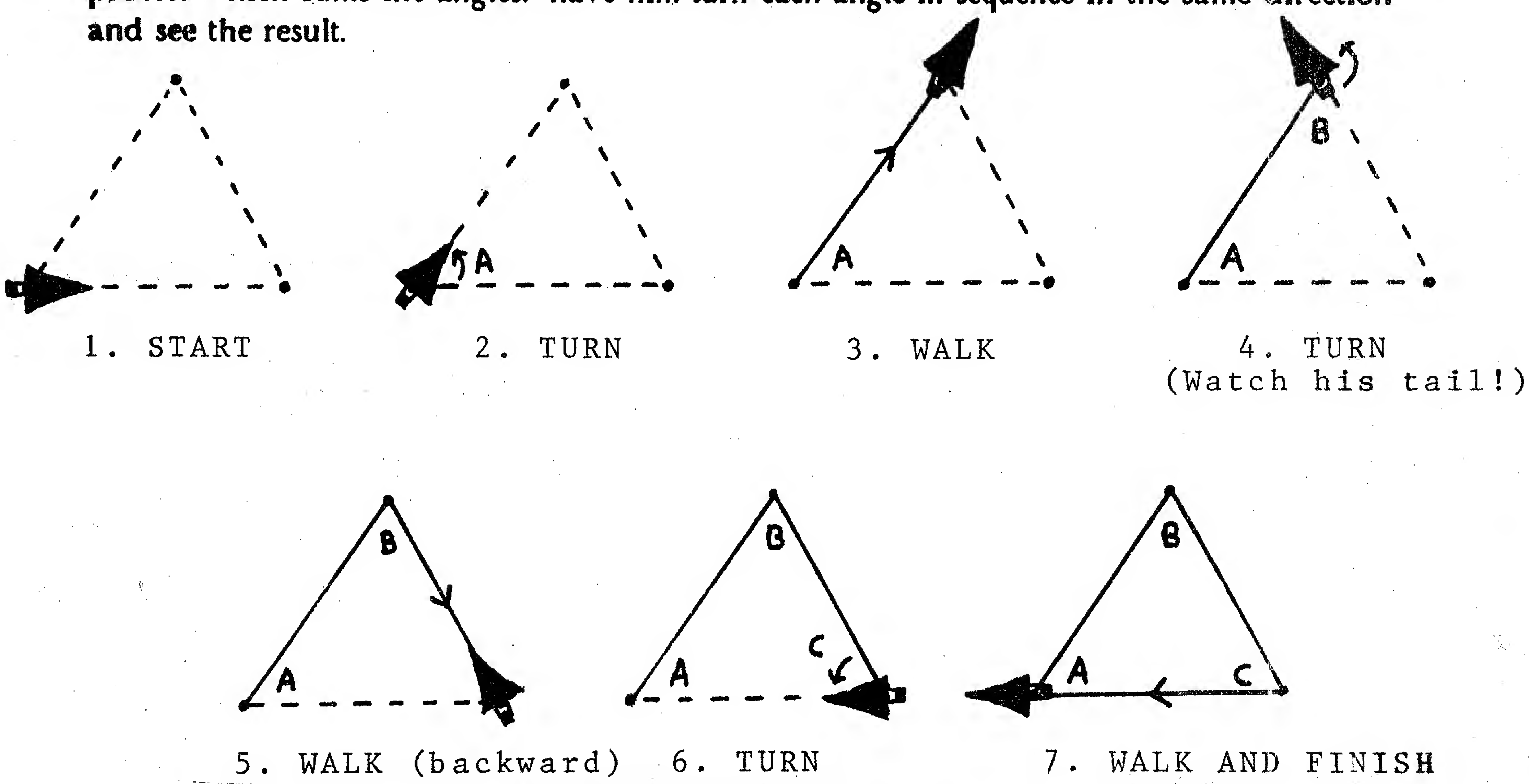


Figure 7. Measuring A + B + C.

Notice the change in the turtle's heading. He's pointing exactly opposite of his initial state and thus has turned by definition 180 degrees.

Figure 7 appears much more complicated than Figure 6 on paper, but consider two important facts. First, the turtle proof as shown is rather clumbsy because it is an

adaptation of a representation which in essence involves motion. Here I must rely on the reader to "make a movie" of that information in his head. With less primitive technology than printing, the essential of the process could be much more easily presented in such a way as to take advantage of human's ability to understand and analyze motion. That ability, of course, accounts for the understandability of the proof in the first place.

Second, the control knowledge which leads to the generation of the proof is as important as the final insight for evaluating comprehensibility. The turtle proof in this respect is transparent, adding angles is merely translated to sequential performance of turns to make a proof. The motivation behind the standard proof is rather more complex.

Whether or not the second method is better than the first, it has some other advantages typical of constructively defined mathematics. Among these is the important property of generalizability: The proofs of similar propositions are, in a natural sense, all the same. The second method needs no modification to work on any polygon (though it is better to measure exterior angles in case the number of sides is greater than four). Consider the turtle process to measure the sum of the exterior angles of a triangle shown in Figure 8.

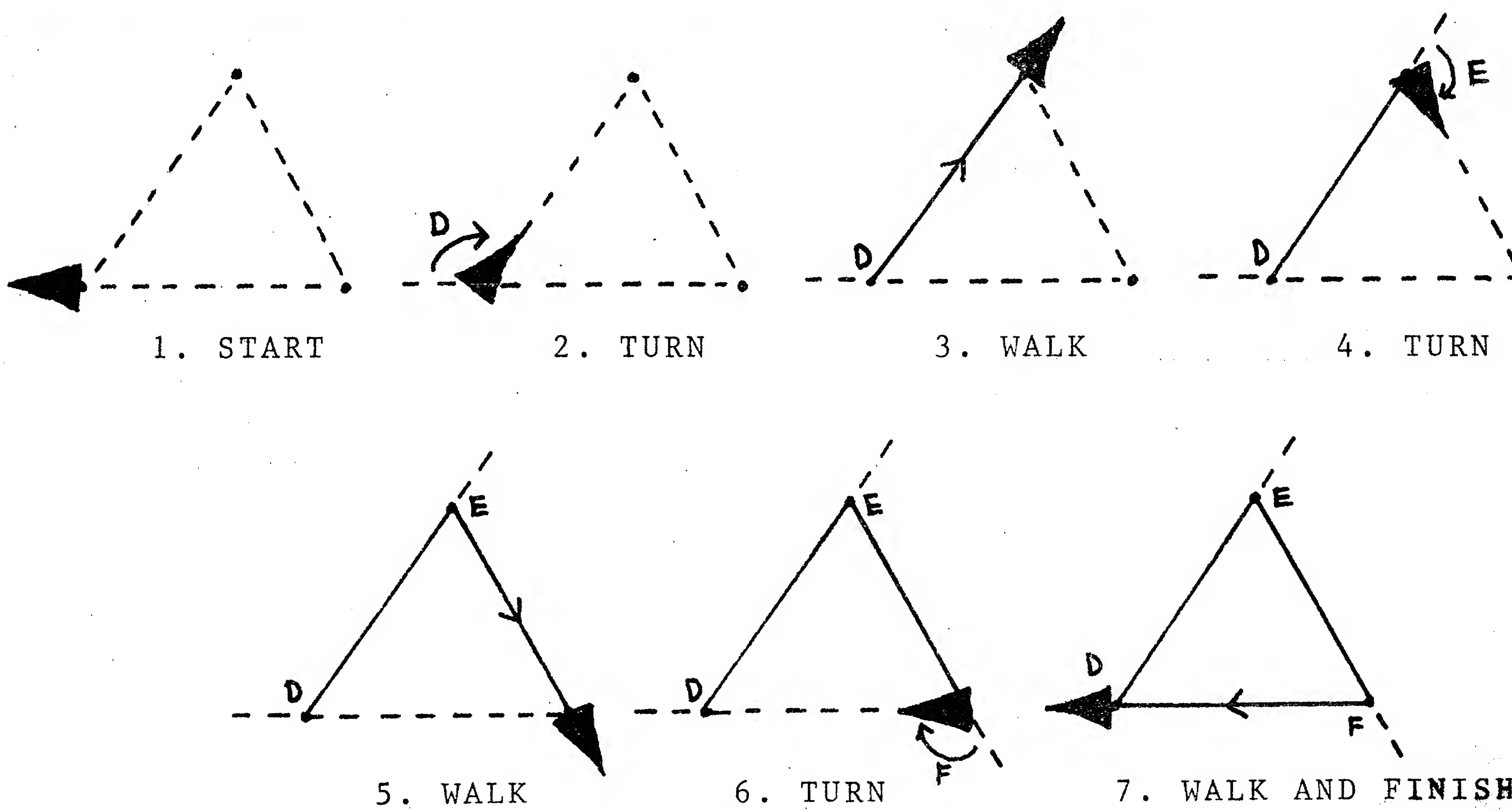


Figure 8. Measuring D + E + F.

The turtle ends his trip with the same orientation as he started. He's turned 360 degrees. Try to make a construction like Figure 6 to demonstrate this. If you succeed at that, notice that the turtle proof depends in no way on the figure being a triangle; it is true of any polygon! The generalization of Figure 6 becomes even less transparent in such circumstances.

Many related problems of angle measurement require only slight modifications of the algorithm. Finally, the method is infallible in the very real sense that one can add up any sequence of angles on any surface in this way, if one is careful. The Euclidean method has nothing to say about triangles on spheres, but the turtle method is a natural lead-in to non-flat geometries. I provide a glimpse of this in example 4. Incidentally, notice the fact that a turtle description is also a prescription for construction. This establishes a vital link from geometry to physics. Motion and trajectories may well be explained in turtle terms. Example 3 shows the power of uniting an active, process oriented geometry with physics.

3 - Orbital Mechanics: A More Complex Example

In order to show how some of these ideas, particularly process oriented representation, can extend beyond introductory concepts I wish to paraphrase a "deduction" of elliptical orbits from the inverse square force law. Naturally I would like to avoid the analytic language of F = ma, but again cannot take the time here to develop a complete, alternate, procedural language. Nonetheless, those who are not familiar with the analytic language should also be able to appreciate the point. The equations are markers for physicists reading this paper, and for the rest will serve to symbolize the lack of illumination a purely analytic presentation can have.

Begin with F = ma

$$\text{(A)} \quad \vec{F} = -k\hat{r}/(r^2) = m\,(d^2\vec{x}/dt^2)$$

where k is a gravitational constant, $\vec{x}$ represents position, r = radius from the sun, $\hat{r}$ is the (unit length) vector pointing radially from the sun to the planet. The first transformation is to shift emphasis from position to velocity. Forces, remember, act directly on velocity and only indirectly on position.

$$-k\hat{r}/(r^2) = m\, d\vec{v}/dt$$

Analytically this appears to be a trivial transformation, but conceptually it is not. The procedure represented by this equation,

$$\vec{v} \leftarrow \vec{v} - (k\hat{r}/r^2 m)\,\Delta t$$

can be easily computer implemented, but one can do much better. If one shifts to looking at r as a function of θ rather than as a function of t, that is, look at the orbit per se rather than the time-parameterized orbit[14], one gets

$$\text{(B)} \quad d\vec{v}/d\theta = -(k/mL)\,\hat{r}.$$

L is a quantity called angular momentum. Now a direct procedural translation (the act of translation is simple but is not important since in practice we would be speaking in procedural terms all along) is the following. Each change in velocity, $\Delta\vec{v}$, has a constant magnitude ($k\Delta\theta/mL$). The change in direction of $\Delta\vec{v}$ between steps is also a constant ($\Delta\theta$). Thus to generate the changing velocity, perform the following algorithm:

(C)
(a) go forward a small amount
(b) turn by a small amount to face a new direction
(c) repeat the above.

If you think for a moment you will realize that that "procedural differential equation" is

precisely the turtle geometric description of a circle.

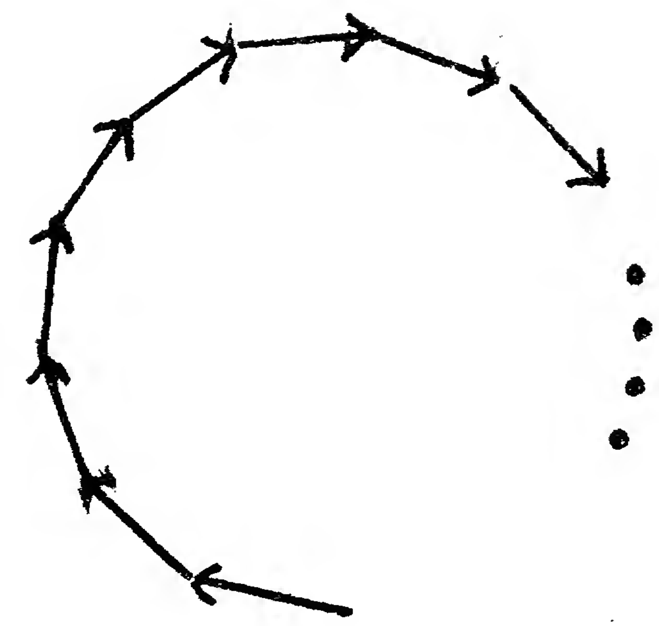


Figure 9. Representation (C), a circle of force impulses (velocity changes).

That such an algorithm generates (a close approximation to) a circle is not a hard thing to understand; give those instructions to an elementary school child and see if he understands that they direct him to walk in a circle!

Knowing the entire sequence of velocities, one can go back to draw conclusions about the position orbit. The orbital problem is in principle solved, and in practice showing such facts as the orbit is an ellipse require only a few algebraic steps.[15]

Solving the problem through the intermediary of velocity is not a special ploy. Velocity is the thing that force changes. Because of that there are in fact great dividends to a velocity oriented derivation. The effect of many perturbations becomes obvious through the intermediary of velocity. Practical and interesting problems such as guidance of an orbiting space ship become immediately accessible.[16] Orbital mechanics can become a domain for student explorations rather than just more results to remember in isolation.

There is no magic in the approach. One must still solve a differential equation. But one can solve it in the form of (A) or (B)[17] which involve a great number of formal operations (check any standard derivation) or in the form of (C). There is no a priori reason to say that representation (C) is simpler than (A) or (B). In fact, it is probably as hard to solve (A) or (B) as to prove that (C) draws a circle. The reason that (C) succeeds in being transparent is that it is phrased in procedural terms which are very close to the knowledge store everyone must have and use to walk around this world.

4 - Turtle Differential Geometry

One great stumbling block to doing non-flat geometry on, say, a high school level is the lack of a good definition for a "line" or more appropriately in the case of non-flat surfaces, a geodesic. The standard definition, a path of shortest or extremal distance, has some

intuitive appeal but is really most appropriate in the analytic context of variational differential equations. A student will probably put up with the shortest distance definition but gets very uneasy when presented with a "longest distance" geodesic like a complete great circle on a sphere. In contrast to this, consider the constructive turtle definition: a geodesic is what a turtle walks if he walks straight, i.e. takes the same number and length of steps with his right and left legs.

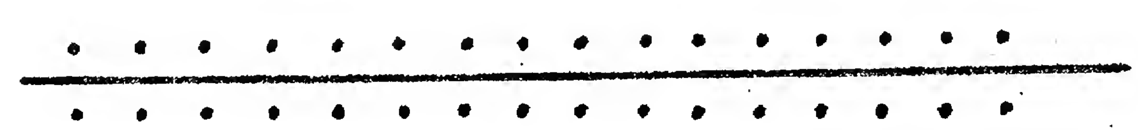


Figure 10. A line with "turtle tracks."

The definition is not only easy to understand, but it is not hard to get high school students to make that definition themselves if encouraged to verbalize about how they can know they are walking a straight line without looking; it is a simple annotation of some knowledge within process.

There are other advantages. The constructed geodesic explicitly mentions the left-right symmetry which is an essential heuristic understanding of "straight." That symmetry shows the equator is a geodesic while an 80 degree latitude cannot be. (The former divides the earth into two equal pieces and the latter does not.) The constructive definition also gives an intrinsic check on whether a path is a geodesic. Does it follow the rules for turtle construction? Can you put an equal number of equally spaced turtle tracks around the "line"?

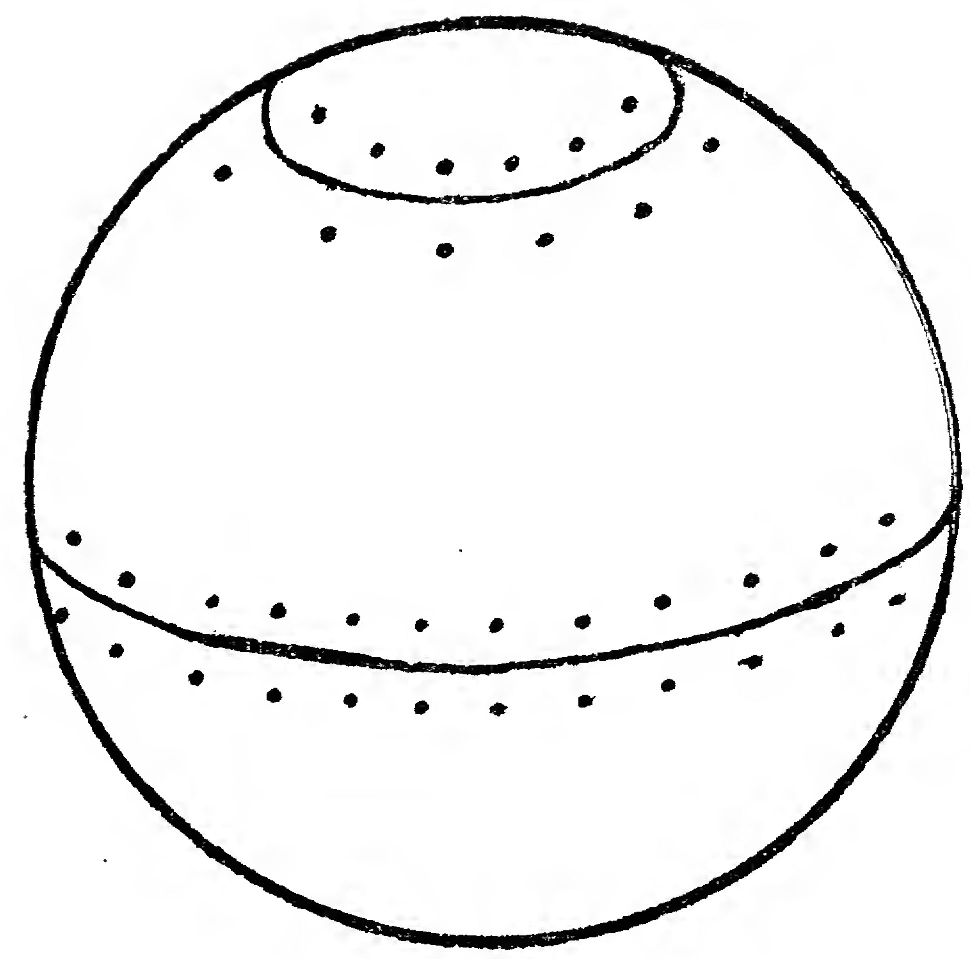


Figure 11. Equator is a "straight" line; other latitudes are not.

Compare this process to proving that there is no shorter (or longer) path, or alternatively, constructing one.

Returning briefly to physics, students often feel quite uneasy with the variationally defined (extremal distance) geodesic. "How can a particle or light ray 'compute' its geodesic path unless it 'knows' its endpoint already?" Granted such questions are confusions, but they arise from feelings for local causality which should be encouraged rather than frustrated. This is exactly the kind of intuitive disposition one wants to take advantage of, not do battle with. Local causality is only a formal property of variational geodesics; it is manifest in the local and constructive turtle definition.[18]

Along the same lines it is very easy to relate a turtle geodesic to such things in everyday experience as the path of a car with wheels straight (each wheel turning at the same speed) or a jet airplane with rudder straight and wing engines running equally fast. The formal elegance of avoiding the question of local construction through a variational definition leaves out these important experiential ties.[19]

Having by-passed formal axiomatics with appropriately active definitions, it is not hard to take this turtle quite far in non-flat geometries[20] -- far enough in fact to bring high school students in contact with many of the most important ideas in mathematics: the concept of transformations and invariants, continuity, the importance of topological considerations, and Stokes-like theorems[21]. It is one of the prime advantages of informal presentations that students can begin developing feelings for and even the ability to use some of these extremely valuable and broadly applicable ideas long before their formal abilities are up to very general and/or precise formulations. Planting the seeds for understanding "powerful ideas" allows time to nurture notions of purpose and use which can keep a student's head above water in the rising tide of details necessary later for true mathematical integrity.

- Retrospective -

I apologize for the rather technical nature of some of these examples. I felt it necessary to cover a broad spectrum of curricula yet give enough detail to allow readers a glimpse of the kind of major overhaul that this paper implies. In addition I hoped to give some sense of how a procedural point of view could provide a coherent backbone to the overhaul, from the most elementary (drawing triangles) to advanced material (orbital mechanics and differential geometry).

VI. Summary

I have argued that axiomatics or other formal systems may be useful models of "good" representations of knowledge for certain purposes, but they are not sufficient as pedagogical models. Other kinds of more informal presentations have a great number of advantages in being able to take into account the specific abilities and knowledge students acquire from everyday experience.

The argument has been organized around the computational metaphor which has two parts.

> a) Human thinking and knowing is process. It is complex but exhibits organization of a type which is hardly of the logical formalist type. A particularly important outgrowth of this is a concern for knowledge which is self-directing, organizational; in short, control knowledge.
>
> b) Because of a) there is good reason to believe that one can produce a significantly more learnable curriculum if one augments the more traditional set of knowledge organization schemes (such as axiomatics) with more procedurally oriented ones. This is particularly true if one can choose procedural representations with elements which match as well as possible the natural knowledge within process which constitutes much of the intuitive, common sense knowledge students already have.

I conclude with an abstracted list of desiderata for pedagogical material.

1) It is a discovery rich environment and is careful to organize the material with many "windows" (not just gaps or holes) for more than exercise type individual study.

2) It discusses and develops "higher level" organizational skills such as heuristics and other control knowledge. In particular, it discusses its own organization and explains the nature of the enterprise with reference to the ultimate goals of the material. The function of ideas in problem solving etc. (qualitative knowledge) is a key part of understanding them.

3) It attempts to access and to tie closely to non-propositional knowledge such as intuition and common sense which students have acquired about the world, whether or not the

knowledge is verbally accessible. Metaphor and analogy have their place in such attempts, not only to explain a structure in approximation, but also to invoke and involve appropriate dispositions and other control structure.

4) In connection with 3), it may well be organized around procedural representations of knowledge, some characteristics of which follow:

a) It is active and constructive rather than prescriptive or descriptive.

b) The large scale unifying form is process rather than deduction.

i) The model form is

$$\text{Initial state} \xrightarrow[\text{Operation}]{} \text{Final state}$$

rather than

$$\text{Assumptions} \xrightarrow[\text{Deduction}]{} \text{Conclusions}$$

ii) Important predicates and relations are: independent (as degrees of freedom in a linear system), invariants, procedural equivalence, etc. rather than true, false, follows from, logical equivalence etc.

5) Presentations are from multiple viewpoints dictated by the use to which the student will put the ideas and what knowledge the student already has which can be accessed. Great care is taken to provide the kind of knowledge which interfaces well with control concerns.

6) In connection with 3) and 5) it may frequently make use of simplified schema with successive corrections and amendments. It is less concerned with the pathological special case except when this is a telling and crucial failure. It is not afraid to introduce "advanced" notions provided they are useful and have intuitive content: By "advanced" I mean ideas which require a large formal background for rigorous "respectable" presentation. Power to understand and to accomplish should be the first lessons of mathematical and scientific knowledge; rigor and precision are secondary.

APPENDIX

Roots

The psychological roots of this paper are in large degree Piagetian. Two of the key ideas in illuminating the computational metaphor are cornerstones of Piagetian psychology.

- Thinking and knowing are interactive processes -

First is the recognition that thinking and knowing are complex highly interactive processes. It is not productive, in this view, to think of human knowledge as a collection of facts or statements any more than education is merely the accretion of such statements. Instead, the active nature of thinking and knowing in their interactions with experience is vital. This active nature is two-fold. (1) It is developmentally dynamic. Experience affects the (cognitive) state of the individual by changing it in ways which are structurally natural developments of the current state. Ideas, facts, theories can only be coherently absorbed into the thinking process if sufficient functional connections between old and new are available. (2) On the other side, understanding is not the mere apprehension of structure in one's experience, but is the molding and transforming of sense experience into forms compatible with the organization internal to the person's thinking and knowing.

The overriding lesson to be learned is that knowledge representations suitable for education (there are other purposes!) must not be constructed with concern solely for the internal structure of the material, but must make allowance for and take advantage of the internal logic, organization and operation, of the student's knowledge and understanding. Piaget explains:

> "If Platonism is right and mathematical entities exist independently of the subject, or if logical positivism is correct in reducing them to a general syntax and semantic, in both cases it would be justifiable to put the emphasis on the simple transmission of the truth by the teacher, that is, the axiomatic language, without worrying too much about the spontaneous ideas of the children. We believe, on the contrary, that there exists a spontaneous and gradual construction of elementary ... structures ... There is, therefore, a body of facts which are, in general, little known to the teacher, but which, once he has a better psychological knowledge, would be of considerable use to him rather than make things more complicated."[22]

In addition, the many Piagetian experiments have begun to show, especially in young children, the nature of the cognitive structures which people develop, and how they are related to each other and interact with experience. Though the details of these discoveries are not of prime importance for me (my concern is at age levels beyond traditional Piagetian experiments) the general flavor undoubtedly has been influential in my judgements about the character of human processing which guided the specific examples in this paper.

- Abstraction vs manipulation -

The second Piagetian cornerstone is indeed about the character of human processing. It is his recognition that procedural knowledge, especially in the form of physical manipulation of self and external objects, plays a vital role in the development of more abstract, verbalizable reasoning.

> "It would seem ... psychologically clear that logic does not arise out of language but from a deeper source and this is to be found in the general coordination of (physical) actions. ... Therefore, it would be a great mistake, particularly in mathematical education, to neglect the role of actions and always remain on the level of language. Particularly with young pupils, activity with objects is indispensable to the comprehension of arithmetical as well as geometric relations."

The natural concomitant of this development is:

> "... the representations or models used should correspond to the natural logic of the levels of the pupils in question, and formalization should be kept for a later moment as a type of systematisation of the notions already acquired. This certainly means the use of intuition before axiomatisation ..."

Piaget's entreaty to work with students as they stand rather than to force them into more formal stages or just giving up until the students have matured is the point. Evidently his feeling for what is mathematical learning goes beyond formal axiomatics into the acquisition of knowledge within process which I outlined. The concern for intuitive formulation in the quotation is symptomatic and is an important general theme in the search for learnable representations of knowledge.

I will give an example of the role manipulation can play in development of abstract thinking which can also illustrate observation and annotation as a vital learning process.[23]

A man recollects at an early age counting a collection of stones in his garden. The then boy lays the stones out, counts them, then rearranges them and counts again, repeating the process as a game. Suddenly it strikes him that he always finds the same number. Puzzling over this unusual coincidence, he realizes, "Of course, I'm doing the same thing each time." But he is not sure; it does seem a bit different with stones in different places. So he modifies the game so that his counting <u>is</u> exactly the same process, each time merely <u>exchanging</u> places of stones and not <u>changing</u> places. Now he sees he really must get the same number since his finger makes exactly the same motion (the counting process is the same) pointing to the sequence of stones (places) each time. Gradually he becomes bolder in his understanding. "So, what if I do move the stones (places) a little bit? Does that really change the counting?" No, indeed. The boy is well on his way toward a very deep understanding about the world, about the process of counting, but at the same time, about the nature of numbers.

- More roots -

The interpretation and development of Piagetian psychology within the computational metaphor has become an ongoing concern in the work of several artificial intelligence researchers. In particular and by way of acknowledgment I mention Seymour Papert who is responsible for many of the threads I have tried to weave together in this paper.

Some figures of historical interest are T. Kuhn, P. Feyerabend, I. Lakatos, and G. Polya who have done much to penetrate to the human core of science and mathematics. Recently, Terry Winograd's work on natural language stirred a great deal of interest in procedural representations and hence helped to create the ambiance which motivated some of my concerns here. Finally, the reader is referred to the works of Moore, Newell, and Simon at Carnegie-Mellon, and Lindsay and Norman et al. at San Diego for other more extensive interpretations of the computational metaphor.[24]

FOOTNOTES

1 It seems quite plausible to regard even the causal interpretation of Ohm's Law not so much as material knowledge but as a model abstracted from usual applications to represent control aspects of the law, i.e. what one does with the entities involved. If common sources of electricity were constant current rather than constant voltage, $E = IR$ might assume the role occupied conventionally by $I = E/R$.

2 With a more restricted sense of the word knowledge, one would almost have to say "generating knowledge" rather than accessing it. Moore and Newell in the quoted fronting to this paper rather forcefully refuse to recognize the difference.

3 The reader may divine here and in what follows that I intend annotation to refer to a very general process of organizing one scheme or piece of knowledge according to the structure of another.

4 This epistemology shares certain insights and motivations with other constructs such as Polanyi's tacit knowledge and Piaget's knowledge in action (compare knowledge within process) and Chomsky's performance-competence dichotemy (compare control + material organization versus material (including knowledge within process) alone).

5 [Courant] p. 79 ff. Parenthetical remark added.

6 [Kline] p. vii.

7 I. Lakatos does an excellent job of pointing out many of the aspects of doing mathematics which are invisible in axiomatic presentations. In particular, he points out how definitions are not just conventions to establish terms, but a vital part of the general strategy of doing mathematics. See [Lakatos].

8 One finds in some axiomatic treatments that one must prove all right angles are equal. Straight angles by nature of the axioms don't need such a proof. Frighteningly one does not go on to prove the seemingly obvious successor theorem that two angles of any particular measure are equal. The logic is of course there, but in the organization of the axiomatic structure -- not in the geometry. Consider further: Is the triangle inequality less fundamental or less "obvious" than some of the axioms found a hundred pages earlier than the proof of this "theorem."

9 Pun intended. I mean human in the sense that the pedagogy is structured to mesh with the character of human information processing. But just as well, I expect and hope from this meshing that the now very often dehumanized relationship of, say, an elementary school student with his arithmetic, can turn into a happier more congenial one.

[10] One should remark that the "genetic" school of pedagogy shares many of the humanizing motivations presented here. But, thought history can teach a good lesson, to mark it the model of "cognitively correct" is making a mistake on the same order as assuming the axioms are the proper synopsis of the history "with the mistakes taken out."

[11] This point is eloquently made in [Papert].

[12] Of course, one may hope to have a propositional framing of a concept which makes explicit usually tacit or implicit implications about context etc. But more ofter one must allow Bridgman's insight as quoted at the beginning of this paper; that what it is, is how you treat it.

[13] The topic of the gyroscope is treated in this intuitive way in [diSessa, "The Gyroscope"]

[14] The step is necessary in any derivation unless one is willing to do elliptic integrals.

[15] Taking the velocity space solution, $\vec{v} = \vec{z} + u\hat{\theta}$ where $\vec{z}$ points to the center of the circle, $u = k/mL$ is the circle's radius (see footnote 17), and multiplying (cross product) by $\vec{r}$, one gets $L = r(u + z \sin\theta)$. Hence $r = L/(u + z \sin\theta)$, the general equation for a conic section in polar coordinates.

[16] See [Abelson, diSessa, Rudolph] and [diSessa, "ORBIT. . ."] for details of this procedural derivation and other discussion.

[17] Actually the velocity equation (B) is already a substantial improvement over (A) even in analytic terms. In standard notation, notice $d\hat{\theta}/d\theta = -\hat{r}$, so that (B) can be written $d/d\theta\{\vec{v} - k\hat{\theta}/mL\} = 0$, hence $\vec{v} = k\hat{\theta}/mL + \vec{z}$ where $\vec{z}$ is a constant. The position space orbit can be trivially derived from this velocity space solution. Functional interpretation of $\vec{z}$ and the other term in the solution given for $\vec{v}$ provide vital links to intuition in this approach to orbital mechanics. See [Abelson, diSessa, Rudolph].

[18] Incidentally, the local-global dichotemy is one of the important heuristic themes from computation which play a central role in turtle geometry and which seem quite valuable in many other areas as well.

[19] A line as an abstract entity with certain properties has been replaced by a line as the process which draws it! One may wish to generalize this comparison of turtle geometry with standard geometry by contrasting mathematics of construction to mathematics of constraint. The latter defines entities by a series of constraints (e.g. axioms), does not deal with the vagaries of models, and does not bother to tell the student either (a) that the entity of concern is a suitable generalization for everything he knows about, or (b) that there are no known examples of such a thing.

[20] [diSessa, "Turtle . . ."] does this in some detail. Extensions will appear shortly.

[21] By the latter I mean any of the group of theorems which compute the totality of something spreading over a region by computing something else on the boundary of that region. Important examples are the fundamental theorem of calculus, the calculus of residues in complex analysis, Gauss's theorem in electrostatics and gravitation, Stokes' Theorem in electrostatics, any conservation law for flowing substances, the concept of state function in thermodynamics, and the existence of potential (e.g. energy) functions.

[22] This and the following quotes are from [Piaget, Comments on Mathematical Education] pp. 69. Underlining added.

[23] The following is a loose rendition of an account given in [Piaget, Gaps in Experience]. Thanks to H. Sinclair for this reference.

[24] "How Can Merlin Understand?" by Moore and Newell in particular lists a number of general issues relating to understanding systems which the reader may wish to compare to those presented here.

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